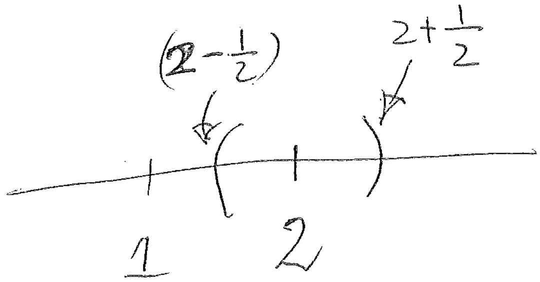
$$\lim_{x \to 2} \frac{x^3-4}{x^2-1} = \frac{4}{3}$$

$$\frac{x^3-4}{x^2-1} - \frac{4}{3} = \frac{3x^3-12-4x^2+4}{3(x^2-1)} = \frac{3x^3-4x^2-8}{3(x^2-1)}$$

$$= \frac{(x-2)(3x^2+2x+4)}{3(x^2-1)}$$



Let $\varepsilon > 0$. Take $\delta > 0$ s.t

$\delta \le 1/2$ (ensuring the image of $V_\delta(2)$ under the demominator map x^2-1 is of strict positive distance to zero)

and $\delta \le \frac{\varepsilon}{37}$ (that is $\delta \le \min\left\{\frac{1}{2}, \frac{\varepsilon}{37}\right\}$)

Let $x \in V_\delta(2)$ $\left(\frac{3}{2} \le 2-\delta < x < 2+\delta < 2+1/2 < 3\right)$. Then

① $|x-2| \le |x|+2 < 5$, δ

② $|3x^2+2x+4| = 3x^2+2x+4 \le 27 + 2 \times 3 + 4 = 37$

③ $|x^2-1| \ge x^2-1 \ge \left(\frac{3}{2}\right)^2 - 1 = \frac{5}{4} \ge 1$

and hence

$$\left|\frac{x^3-4}{x^2-1} - \frac{4}{3}\right| \le \frac{|x-2|\,|3x^2+2x+4|}{3\,|x^2-1|} < \frac{1}{1} \times 37 \times \delta \le \varepsilon$$

if $\delta \le \frac{\varepsilon}{37}$

Chain Rule (Change of Variations)

$A \xrightarrow{g} B \xrightarrow{f} \mathbb{R}$

$x_0 \in A^c$, $y_0 = \lim_{x \to x_0} g(x) \in B^c$, $\lim_{y \to y_0} f(y) = l$

$$\overset{?}{\Longrightarrow} \lim_{x \to x_0} f(g(x)) = l = \lim_{y \to \lim_{x \to x_0} g(x)} f(y)$$

Yes, if at least one of the following conditions is satisfied:

(a) $\exists \gamma_0 > 0$ s.t. $g(x) \neq y_0 \ \forall x \in V_{\gamma_0}(x_0) \cap (A \setminus \{x_0\})$;

(b) $y_0 \in B$ and $f(y_0) = l$.

Proof. Let $\varepsilon > 0$. Then $\exists \delta > 0$ s.t.

(1) $|f(y) - l| < \varepsilon$ whenever $y \in B \setminus \{y_0\}$ and $0 < |y - y_0| < \delta$

For this $\delta > 0$ $\exists \gamma \in (0, \gamma_0]$ (in the case of (a))

(2) $|g(x) - y_0| < \delta$ whenever $x \in (A \setminus \{x_0\}) \cap V_\gamma(x_0)$

Combining (1) and (2) and providing that (a) or (b) holds, we have (Why?)

$$|f(g(x)) - l| < \varepsilon \text{ whenever } x \in (A \setminus \{x_0\}) \cap (V_\gamma(x_0))$$

If (a) also holds then (2) can be rewritten as

(2a) $0 < |g(x) - y_0| < \delta$ whenever $x \in (A \setminus \{x_0\}) \cap V_\gamma(x_0)$

If (b) holds (rather than (a)) then (1) can be rewritten

(1b) $|f(y) - l| < \varepsilon$ whenever $y \in B$ and $|y - y_0| < \delta$

Proof of Product Rule for Limits.

By the local boundness theo, $\exists\, \delta' > 0$, and $M > 0$ s.t

$|f_i(x)| \leq M \;\forall x \in V_{\delta'}(x_0) \cap (A \setminus \{x_0\})$. W.l.g we can take $M > |l_1|, |l_2|$. Let $\varepsilon > 0$. Take $\varepsilon_1 = \varepsilon_2 = \frac{\varepsilon}{2M}$.

Then $\exists\, \delta_1, \delta_2 > 0$ s.t. with each $i = 1, 2$

$$|f_i(x) - l_i| < \varepsilon_i \quad \forall\, x \in V_{\delta_i}(x_0) \cap (A \setminus \{x_0\}).$$

Let $\delta = \min\{\delta_1, \delta_2, \delta'\}\ (>0)$. Claim that

$$|f_1(x) f_2(x) - l_1 l_2| < \varepsilon \quad \forall\, x \in V_\delta(x_0) \cap (A \setminus \{x_0\}).$$

Indeed, let $x \in V_\delta(x_0) \cap (A \setminus \{x_0\})$. Then

$$|f_1(x) f_2(x) - l_1 l_2| \leq |f_2(x)|\,|f_1(x) - l_1| + |l_1|\,|f_2(x) - l_2|$$

$$\lneq M\varepsilon_1 + M\varepsilon_2 = \varepsilon, \text{ as claimed}$$

2nd method (not to use any theorem for limits: direct from definition).

Let $\varepsilon > 0$ and take $\varepsilon' = \min\{1, \varepsilon\}$. Let

$$\varepsilon_1 = \frac{\varepsilon}{2(|l_2|+1)} \quad \text{and} \quad \varepsilon_2 = \frac{\varepsilon}{2(|l_1|+1)}$$

Since $\lim f_i(x) = l_i$, $\exists\, \delta_i > 0$ s.t.

$$|f_i(x) - l_i| < \varepsilon_i \quad \forall\, x \in V_{\delta_i}(x_0) \cap (A \setminus \{x_0\})$$

(noting $\varepsilon_i \leq 1$ it follows from Δ-ineq that $|f_i(x)| < |l_i| + 1$). Then, $\forall\, x \in V_{\delta_1 \wedge \delta_2}(x_0) \cap (A \setminus \{x_0\})$, one has

$$|f_1(x)f_2(x) - l_1 l_2| \le |f_2(x)|\,|f_1(x) - l_1| + |l_1|\,|f_2(x) - l_2|$$
$$< (|l_2|+1)\,\varepsilon_1 + |l_1|\,\varepsilon_2 = \varepsilon.$$

Proof of Quotient Computation Rule. Take δ', $M>0$ as in the Proof for Product. Also, since $\lim_{x\to x_0}|f_2(x)| = |l_2| \neq 0$ and $\frac{1}{2}|l_2| < |l_2| < \frac{3}{2}|l_2|$, we apply the Order-Preserving Th to get $\delta''>0$ such that

$$\frac{|l_2|}{2} < |f_2(x)| \quad \forall\, x \in V_{\delta''}(x_0) \cap (A \setminus \{x_0\}).$$

Let $\varepsilon>0$, and let $\varepsilon_1, \varepsilon_2 > 0$ be defined by

$$\varepsilon_1 = \frac{|l_2|^2\,\varepsilon}{4(|l_2|+1)} \quad \text{and} \quad \varepsilon_2 = \frac{|l_2|^2\,\varepsilon}{4(|l_1|+1)}$$

Take $\delta_1, \delta_2 > 0$ accordingly as in our 1st proof for Product. Let $\delta = \delta' \wedge \delta'' \wedge \delta_1 \wedge \delta_2$: $\delta = \min\{\delta', \delta'', \delta_1, \delta_2\}$.

Then

$$\left|\frac{f_1(x)}{f_2(x)} - \frac{l_1}{l_2}\right| < \varepsilon \text{ whenever } x \in V_\delta(x_0) \cap (A \setminus \{x_0\}).$$

Indeed, if $x \in V_\delta(x_0) \cap (A\setminus\{x_0\})$, then

$$\left|\frac{f_1(x)}{f_2(x)} - \frac{l_1}{l_2}\right| = \frac{|f_1(x) l_2 - f_2(x) l_1 - l_1 l_2 + l_1 l_2|}{|f_2(x)|\,|l_2|}$$
$$\le \frac{|l_2|\,|f_1(x) - l_1| + |l_1|\,|f_2(x) - l_2|}{|l_2|/2 \cdot |l_2|}$$
$$< \frac{(1+|l_2|)\,\varepsilon_1 + (1+|l_1|)\,\varepsilon_2}{|l_2|^2/2} = \varepsilon.$$

(Remark. For the denominator in ε_1 you can use $|l_2|$ in place of $|l_2|+1$ but for that in ε_2 you cannot use $|l_1|$ in place of $|l_1|+1$ as l_1 may be zero.)

2nd Proof for Quotient Rule (not use any limit results). Let $\varepsilon > 0$. Let $\varepsilon_1, \varepsilon_2 > 0$ be defined by

$$\varepsilon_1 = \frac{|l_2|\,\varepsilon}{4}, \qquad \varepsilon_2 = \min\left\{\frac{|l_2|}{2}, \frac{|l_2|^2}{4(|l_1|+1)}\right\}$$

Since $\lim_{x \to x_0} f_i(x) = l_i$, $\exists\ \delta_i > 0$ s.t.

$$|f_i(x) - l_i| < \varepsilon_i \quad \forall\ x \in V_{\delta_i}(x_0) \cap (A \setminus \{x_0\}).$$

Then

$$\left|\frac{f_1(x)}{f_2(x)} - \frac{l_1}{l_2}\right| < \varepsilon \quad \forall\ x \in V_{\delta_1 \wedge \delta_2}(x_0) \cap (A \setminus \{x_0\}).$$

Indeed, if $x \in V_{\delta_1 \wedge \delta_2}(x_0) \cap (A \setminus \{x_0\})$, one has $-|f_2(x)| + |l_2| \leq |l_2 - f_2(x)| < \varepsilon_2 \leq \frac{|l_2|}{2}$ and

$$\left|\frac{f_1(x)}{f_2(x)} - \frac{l_1}{l_2}\right| \leq \frac{|l_2|\,|f_1(x) - l_1| + |l_1|\,|f_2(x) - l_2|}{|f_2(x)| \cdot |l_2|}$$

$$\leq \frac{|f_1(x) - l_1|}{|f_2(x)|} + \frac{|l_1|\,|f_2(x) - l_2|}{|f_2(x)| \cdot |l_2|}$$

$$< \frac{2}{|l_2|} \cdot \varepsilon_1 + \frac{|l_1|}{|l_2|^2/2}\,|f_2(x) - l_2| \leq \frac{\varepsilon}{2} + \frac{|l_1|}{|l_2|/2}\,\varepsilon_2 \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$